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Reflection of electromagnetic waves.

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Reflection of Electromagnetic Waves

by

Joseph B. Keller

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ABSTRACT

In a homogeneous, isotropic, non-conducting medium there is placed a source of radiation which varies harmonically with time, and an obstacle which is considered to be perfectly reflecting. The reflected field at any point is then determined approximately by the Kirchoff method. As a result it is found that the reflection coefficient (ratio of reflected amplitude to incident amplitude) is made up of two factors - a geometrical factor and a phase factor. The geometrical factor is also evaluated by geometrical optics. The special form of this factor is determined for reflection back to the source from an arbitrary surface, and in particular the results are given for reflection from a plane, a cylinder, and a sphere.

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REFLECTION OF ELECTROMAGNETIC WAVES*

1. Introduction. This paper is a partial study of the problem of reflection of electromagnetic waves from an obstacle. Some work on this subject was done during the war, and it is believed that the present paper extends the results obtained and furnishes a new approach not hitherto fully exploited. The basic method used is what is commonly known as the Kirchoff method, and an essential feature of the result is the expression of the reflection coefficient (ratio of reflected amplitude to incident amplitude) as the product of two factors, each associated with certain special characteristics of the problem.

In order to determine the wave reflected from a target or obstacle, it is necessary to obtain solutions of Maxwell's equations subject to certain boundary conditions. It is usual to assume that the field is known in the absence of the obstacle, and that the sources behave in the same way whether or not the obstacle is present. The latter assumption, which is always valid when the source emits a single short pulse, seems to be valid even in the case of a harmonic source except when the obstacle is very near the source; that is, in general the presence of the obstacle does not significantly alter the radiation impedance of the source. It is also convenient to assume that the medium, in addition to be isotropic and nonconducting, is homogeneous. While this assumption is not strictly valid, the usual inhomogeneity is small enough to permit separate calculations of the reflected wave amplitude and of its propagation. It is also simplest to assume that the medium is infinite. If this is done, it is then possible to

* This paper makes use of some of the methods and results of a study with which the author was associated, on the reflection of sound waves, viz. "Reflection and Transmission of Sound by Thin Curved Shells" by Henry Primakoff, Washington University, St. Louis, Missouri, and Joseph B. Keller, Institute for Mathematics and Mechanics, New York University, New York, N. Y.

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take account of additional obstacles or boundary surfaces by considering reflection from the boundary surfaces of the wave reflected from the target (Multiple reflection). An alternative procedure involves taking the boundary surfaces into account from the beginning. Except for the simplest boundary shapes and boundary conditions, this method is much more difficult than the former, and very few problems have been solved in this way; Therefore this procedure will not be followed here. In addition, the sources are assumed to vary harmonically; results for other time dependences can be obtained by Fourier analysis.

Under the above assumptions, each component (in rectangular coordinates) of electric and magnetic field satisfies the wave equation in the medium outside the sources and target. Therefore methods applicable to the solution of the wave equation, and hence of acoustic problems, can be used.

In the case of the acoustic problem mentioned in the footnote on Page 2. when the reflecting object is a thin shell, we find that within the limits of the approximations used the reflected pressure is equal to the incidant pressure multiplied by three factors. One of these is associated with geometrical features including the curvature of the reflecting surface, distances of the source and of the observation point, etc., and may be called the geometrical factor. A second, called the phase factor, depends upon the dimensions of the obstacle, the wave length, etc., and accounts for the interference of the wavelets reflected from different parts of the surface. The third factor, depending upon the difference in the physical properties of the shell and the medium and also upon the ratio of the thickness of the shell to the wave length, accounts for the fraction of the energy (or of the amplitude) which is reflected. This factor may be called the reflectivity factor.

The general method by which these factors are found for sound waves cannot be carried over without change to the reflection of radio waves. That which makes the radio problem essentially different from the sound problem, however, seems to be isolated in that part of the problem that has to do with the reflectivity factor and it seems probable that changes in the general procedure can be made which will make it possible to solve this problem as completely as the sound problem has been solved. In the paper on sound it is shown that the phase factor and the geometrical factor obtained by the general method can also be obtained by the Kirchoff method, which is also applicable in the electromagnetic case. Moreover, the geometrical factor can be obtained, both for sound waves and for radio waves, by geometrical optics. The application of these methods to the reflection of radio waves is made in the present paper (the method of geometrical optics is in the appendix).

It is the essence of the Kirchoff method that the wave equation is transformed into an integral equation which is then reduced by means of assuming certain boundary conditions to the problem of evaluating an integral. While the assumptions made concerning boundary values are not altogether justifiable, nevertheless reasonably accurate results have been obtained by this method. In the case of the present paper it should be pointed out that the result obtained by the Kirchoff method approaches that of geometrical optics as the wave length approaches zero. In particular, the law of reflection, the mirror law, and the condition for the formation of point images are obtained. When the method is applied to refraction, Snell's law, the lens law, and the condition for point images are also obtained. In addition the 180° change of phase at a focus is observed. These facts furnish partial justification of the approximations made in calculating the result and are evidence of the propriety of the assumptions made.

For a complete solution of the reflection problem the physical properties of the reflecting body must be taken into account, as well as its thickness as compared to the wave length. If there is perfect reflectivity, however, i.e. all incident energy is reflected, only the surface of the body need be considered. In such cases the reflection coefficient consists of the phase factor and the geometrical factor only. The determination of these factors in this paper therefore gives the approximate reflection coefficient for any perfectly reflecting body, and the approximation should be of practical value even when the reflectivity of the body is somewhat less than peffect.

The value of the results obtained in this paper is two-fold: (1) The present results will serve as a check on any results that purport to give the complete reflection coefficient, since this coefficient should include the phase and geometrical factors here obtained and reduce to these alone when the properties of the obstacle are such as to make the reflectivity perfect. (2) Since many bodies are of such material that their physical properties have little effect on reflection, and hence have nearly perfect reflectivity, the results obtained will be good enough approximations for many practical cases.

Several papers have been written on this subject, and it would be desirable that they be compared for the purpose of evaluating the methods used and the results obtained. Moreover, it is the intention of the present writer to attack the problem from the more general viewpoint mentioned above, taking into consideration all the factors affecting the reflection coefficient.

2. Solution by the Kirchoff Method. The problem to be solved is the following: We assume, as outlined in the Introduction, a medium that is homogeneous, isotropic, and non-conducting, in which is placed a harmonic source of radiation producing a known field. In this medium there is placed an obstacle from which reflection can be expected. We have then to determine the reflected field at any point.

At any point in the medium outside the source and the obstacle each component of the electric field and of the magnetic field satisfies the wave equation

$$\left(\nabla^2_{\vec{r}} + k^2\right) P(\vec{r}) = 0. \qquad (1)$$

In this equation $\vec{r}$ is the position vector of any point, $P(\vec{r})$ represents any component of the electric or magnetic field, $\nabla^2_{\vec{r}}$ is the Laplacian operator, and $k = \frac{2\pi}{\lambda}$.

By the Kirchoff-Helholtz theorem*, $P(\vec{r})$ also satisfies the integral equation

$$P(\vec{r}) = P_0(\vec{r}) + \frac{1}{4\pi}\int_S \left\{\frac{\partial P(\vec{r}')}{\partial n'} G(\vec{r},\vec{r}') - P(\vec{r}')\frac{\partial G(r,r')}{\partial n'}\right\} ds \qquad (2)$$

for $\vec{r}$ not on S. Here $P_0(\vec{r}')$ is the field component with the obstacle absent, S is the surface of the obstacle, $\vec{r}'$ refers to any point on the surface and $G(r,r') \equiv \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$.

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* Baker, Bevan B., and E. T. Copson: The Mathematical Theory of Huygen's Principle. p. 24.

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The characteristic feature of the Kirchoff method consists in assigning boundary values to $f(r')$ and $\frac{\partial P(\vec{r'})}{\partial n'}$ on S so as to reduce the problem of solving equation (2) to a problem of evaluation of a surface integral. We assume that $P = P_0$ on the geometrically illuminated part of S, $P = 0$ on the remainder of S, and $\frac{\partial P}{\partial n'} = 0$ on all of S. These boundary values are chosen in accordance with the usual practice in diffraction problems. They may be thought of as applying to a perfect reflector and, on the basis of geometrical optics, do not appear unreasonable.

Inserting these boundary values into equation (2), we have

$$P(\vec{r}) = P_0(\vec{r}) - \frac{1}{2\pi}\int_{S'} P_0(\vec{r'}) \frac{\partial \Xi(\vec{r},\vec{r'})}{\partial n'} \, da, \tag{3}$$

where S' is the illuminated part of S. In order to simplify the integral further let us assume that $|r - r'| >> \lambda$. Then, if α denotes the angle between $\vec{r'} - \vec{r}$ and $\vec{n'}$,

$$\frac{\partial \Xi}{\partial n'} = \Xi \cos\alpha \left(ik - \frac{1}{|r - r'|} \right) \approx ik \, \Xi \cos\alpha, \tag{4}$$

and $$P(\vec{r}) \approx P_0(\vec{r}) - \frac{ik}{2\pi} \int_{S'} P_0(\vec{r'}) \, \Xi(\vec{r'}, \vec{r}) \cos\alpha \, da \, . \tag{5}$$

It will be assumed that the source is such that

$$P_0(\vec{r'}) = A \frac{e^{ik|r' - r_0|}}{|r' - r_0|},$$

where A may depend upon the angle between $\vec{r'} - \vec{r_0}$ and some fixed direction. This is the form of the field due to an oscillating dipole at distances large compared to λ, and is also approximately correct at large distances from other antennas.

Therefore

$$P(\vec{r}) \approx P_0(\vec{r}) - \frac{ik}{2\pi} \int_{S'} A \cos\alpha \, \frac{e^{ik\left\{k|r' - r_0| + k|r' - r_1|\right\}}}{|r' - r_0|\,|r' - r|} \, da \, . \tag{7}$$

This integral is evaluated approximately in Appendix I and the result is

$$P(\vec{r}) \approx P_o(\vec{r}) + \frac{A}{D_1}\left[\frac{k\cos}{\pi\omega_2^2}\right]\left[\frac{e^{iV'}}{i}\left\{F(x_o') - F(x_o'')\right\}\left\{F(y'_o) - F(y''_o)\right\}\right] \qquad (8)$$

where D_1, D_2 are the average distances from the surface to the point source and point of observation, respectively, V' is a phase factor defined in Appendix I, $F(x)$ is the Fresnel integral function of x, and x_o', x_o'', y_o', y_o'' are the limits of integration on the approximately rectangular "illuminated" region of the surface. Furthermore

$$J = \frac{k_a}{\pi}\left(\left(\frac{\cos\alpha}{D_2}\right)^2 + \frac{1}{D_2} + \left(\frac{1}{D_1}\right.\right.\left[2 - \sin^2\gamma - \sin^2\alpha + 2\sin^2\alpha\sin^2\gamma\sin^2(\theta-\phi)\right]$$

$$\left. - \left[\cos\alpha + \cos\gamma\right]\left[2G_m\cos^2\alpha + G_{11}(\theta)\sin^2\alpha\right]\right)$$

$$+ \left(\frac{\cos\gamma}{D_1}\right)^2 + G_s(\cos\alpha + \cos\gamma)^2$$

$$- \frac{1}{D_1}(\cos\alpha + \cos\gamma)\left(2G_m\cos^2\gamma + G_{11}(\phi)\sin^2\gamma\right)^{1/2}$$

where

γ = angle of incidence = average angle between $\vec{r}' - \vec{r}_0$ and $\vec{n}'$

α = angle of reflection = angle between $\vec{r}' - \vec{r}$ and $\vec{n}'$

ϕ = angle between plane of incidence and normal plane through section of principal curvature C_1

θ = angle between plane of reflection and normal plane through section of principal curvature C_1

C_1, C_2 = principal curvatures of surface

$C_m = \frac{C_1 + C_2}{2}$ = mean curvature of surface

C_g = Gaussian curvature of surface

$C_{\parallel}(\theta)$, $C_{\parallel}(\phi)$ = Curvatures of surface in planes of reflection and incidence, respectively

(These quantities all refer to the point on the surface at which the phase in the integrand of eq. (7) is stationary. See Appendix I.)

From equation (8) it is seen that the amplitude of the reflected wave, $p(\vec{r}) = P(\vec{r}) - P_0(\vec{r})$, is given by $\frac{A}{D}$, the amplitude of the incident wave at the obstacle, multiplied by a certain "reflection coefficient". The reflection coefficient for amplitude is defined as the ratio of the amplitude of the reflected wave at $\vec{r}$ to the incident wave at the obstacle. It is seen in equation (8) to consist of a geometrical factor and a phase factor.

The reflection coefficient is maximum in directions of specular reflection (angle of incidence = angle of reflection), and as the wave length tends to 0 (k infinite) the coefficient becomes zero except in directions of specular reflection. For specular reflection $\alpha = \gamma$ and $\theta = \phi$, and the reflection coefficient is

$$\left|\frac{p\ (r)}{P_0(\text{surface})}\right| = \left[1 + 2D_2\left\{\frac{1}{D_1} - 2G_{\perp}\cos\alpha - G_{\parallel}\sin\alpha\tan\alpha\right\} + D_2^2\left\{\frac{1}{D_1^2} - \frac{4G_{\perp}\cos\alpha}{D_1} + 4G_{\perp}G_{\parallel} - \frac{\partial G_{\parallel}\sin\alpha\tan\alpha}{D_1}\right\}\right]^{-1/2}$$

$$\cdot \left|\frac{\left\{F(x_0') - F(x_0'')\right\}\left\{F(y_0') - F(y_0'')\right\}}{2}\right| \qquad (9)$$

If the Fresnel integrals (in the phase factor) have their asymptotic values, as is always the case for sufficiently small wave lengths, the phase factor equals one and the reflection coefficient reduces to the geometrical factor, which is the same as the reflection coefficient obtained by geometrical optics (Appendix II).

When the source and observation points coincide $D_1 = D_2 \equiv D$. If the reflection is specular, $\alpha = 0$ since the reflection must be normal. Then the geometrical factor G assumes the simple form

$$G = \frac{1}{2}\left(1+2G_m D+G_g D^2 \right)^{-1/2} . \qquad (10)$$

(In this expression the sign of G_m has been changed so that a curvature is positive if the center of curvature is on the opposite side of the obstacle from the source.) In special cases this yields:

Plane: $G = \frac{1}{2}$.

Cylinder: $G = \frac{1}{2(1+\frac{D}{R})}$, R = radius.

Sphere: $G = \frac{1}{2(1+\frac{D}{R})^2}$, R = radius.

If D^2 is so much larger than the radii of curvature that $G_g D^2 >> 2G_m D+1$, one has approximately $G = \frac{1}{2D\sqrt{G_g}}$.

The scattering cross section σ is defined by

$\sigma = 4\pi D^2 R^2$, where R is the amplitude reflection coefficient and as above $D_1 = D_2 \equiv D$. When the phase factor equals one, $R = G$ and for the case of specular reflection

$$\sigma = \frac{\pi D^2}{(1+2G_m D+G_g D^2}$$.

For D much larger than the radii of curvature this becomes

$$\sigma = \frac{\pi}{G_g} = R_1 R_2 ,$$

where R_1 and R_2 are the principal radii of curvature of the surface.*

*A different result (obtained by geometrical optics) was given by P.M.Marcus, Radiation Laboratory Report 1022. However, an examination of his paper reveals the presence of unjustified approximations in the fundamental equation for the reflected ray.

The reflection coefficient can become infinite when J in the denominator of equation 8 vanishes, but in this case, as mentioned in Appendix I, the evaluation of the integral is not valid. It is then necessary to retain third order terms in the phase of the integrand. An investigation of the points at which $J = 0$, however, leads to an expression for the focal length of a curved mirror, the necessary conditions for the occurrence of point images, the mirror law and the 180^0 change of phase at a focus. These considerations will not be discussed farther here.

For specular reflection when the illuminated part of the surface is symmetric about the point of specular reflection, the phase factor of equation 9 becomes $2\left|F(x_0') \cdot F(y_0')\right|$.

Here the source and observation point are assumed to coincide and x_0' and y_0' have the following values:

$$x_0' = 2\chi'\sqrt{\frac{1}{\lambda D}\left(1+\frac{D}{R}\right)}$$

$$y_0' = 2\eta'\sqrt{\frac{1}{\lambda D}\left(1+\frac{D}{R_2}\right)} \quad .$$

where R_1 and R_2 are the negatives of the principal radii of curvature of the surface. The square roots are just the half-dimensions of the first Fresnel zone on the surface, so x_0' is twice the ratio of the length of the illuminated part of the surface to the length of the first Fresnel zone, and similarly for y_0'. Thus when the surface subtends many Fresnel zones at the source, x_0' and y_0' are large and the Fresnel integrals approach their asymptotic values. Then the phase factor approaches one. On the other hand, when x_0' and y_0' are small, and the surface subtends only part of a Fresnel zone, the Fresnel integrals may be replaced by their arguments and the phase factor becomes

$$8\,\chi'\eta'/\lambda D\left(1+\frac{2}{R_1}\right)^{-1/2}\left(1+\frac{D}{R_2}\right)^{-1/2}$$

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which is 8 times the area of the (illuminated) surface divided by the area of the first Fresnel zone.

The dimensions of the first Fresnel zone are small and x_0', y_0' large unless either or both of the radii of curvature are much larger than D, as is the case for a plane or cylindrical reflector. If, for example, R_1 is infinite, then $x_0' = \frac{2\lambda'}{\sqrt{\lambda D}}$. Since D is usually large x_0' may be small, which will make the reflection coefficient small. Except for such surfacesthe Fresnel integrals will usually attain their asymptotic values and the phase factor will then equal one. Finally, the phase factor is much smaller than one in directions other than those of specular reflection.

Appendix I

Evaluation of the Surface Integral

The integral to be evaluated is of the type

$$I = \int_S f(\vec{r}') \frac{e^{i(k_1|\vec{r}' - \vec{r}_1| + k_2|\vec{r}' - \vec{r}_2|)}}{|\vec{r}' - \vec{r}_1|\,|\vec{r}' - \vec{r}_2|}\, da, \qquad (11)$$

where $f(r')$ is a slowly varying function of r'. Since $\dfrac{f(\vec{r}')}{|\vec{r}' - \vec{r}_1|\,|\vec{r}' - \vec{r}_2|}$ is slowly varying, it will be taken out of the integral and replaced by its average value $\dfrac{\overline{f}}{D_1 D_2}$. D_1 and D_2 are the average values of $|\vec{r}' - \vec{r}_1|$ and $|\vec{r}' - \vec{r}_2|$ respectively. In addition the expressions for $|\vec{r}' - \vec{r}_1|$ and $|\vec{r}' - \vec{r}_2|$ in the exponent will be expanded in a Cartesian coordinate system with origin on the surface S and the z' axis normal to S. The direction of the x' and y' axes is chosen so that the equation of S is $z' = a(x')^2 + b(y')^2 \ldots$ without a term in $x'y'$. Then one has (dropping the primes)

$$|\vec{r} - \vec{r}_1| = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}\ .$$

Expanding this function of x, y, and z in the neighborhood of (x_1, y_1, z_1) we have, to terms of second degree in x and y,

$$|\vec{r} - \vec{r}_1| = D_1 - \frac{x_1 x}{D_1} - \frac{y_1 y}{D_1} + \frac{\frac{1}{2} - a z_1 - \frac{1}{2}\left(\frac{x_1}{D_1}\right)^2}{D_1} x^2$$

$$+ \frac{\frac{1}{2} - b z_1 - \frac{1}{2}\left(\frac{y_1}{D_1}\right)^2}{D_1} y^2 - \frac{x_1 y_1}{D_1^2} xy \Big)$$

and a similar expression for $\left|\vec{r} - \vec{r}_2\right|$. Inserting these expressions in the integral gives

$$I \approx \frac{\bar{F}}{D_1 D_2} \int_S \exp\left[i(Ax^2 + Bxy + Cy^2 + Dx + Ey + F)\right] dx\, dy \qquad (12)$$

where $A = \frac{k_1}{D_1}\left(\frac{1}{2} - ax_1 - \frac{1}{2}\frac{x_1^2}{D_1^2}\right) + \frac{k_2}{D_2}\left(\frac{1}{2} - ax_2 - \frac{1}{2}\frac{x_2^2}{D_2^2}\right)$,

$C = \frac{k_1}{D_1}\left(\frac{1}{2} - by_1 - \frac{1}{2}\frac{y_1^2}{D_1^2}\right) + \frac{k_2}{D_2}\left(\frac{1}{2} - by_2 - \frac{1}{2}\frac{y_2^2}{D_2^2}\right)$,

$B = -\frac{k_1 x_1 y_1}{D_1^3} - \frac{k_2 x_2 y_2}{D_2^3}$, $\qquad F = k_1 D_1 + k_2 D_2$,

$D = -\frac{k_1 x_1}{D_1} - \frac{k_2 x_2}{D_2}$, $\qquad E = -\frac{k_1 y_1}{D_1} - \frac{k_2 y_2}{D_2}$.

Let

$$x' = \sqrt{\frac{2}{\pi}\left|A'\right|}\,(x\cos\theta + y\sin\theta + H),$$

$$y' = \sqrt{\frac{2}{\pi}\left|C'\right|}\,(-x\sin\theta + y\cos\theta + K),$$

where $\tan 2\theta = \frac{B}{A - C}$, $A' = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta$,

$C' = A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta$.

Also let

$$H = \frac{D\cos\theta + E\sin\theta}{2A'}, \quad K = \frac{E\cos\theta - D\sin\theta}{2C'},$$

and $F' = A'H^2 + C'K^2 + D'H + E'K + F'$.

now

$$J = \frac{\partial(x', y')}{\partial(x, y)} = \frac{2}{\pi}\sqrt{|A'C'|} = \frac{2}{\pi}\sqrt{\left|AC - \frac{B^2}{4}\right|} \qquad (13)$$

$$= \left| \frac{2}{\pi} \left[\frac{K_1}{D_1} \left(\frac{1}{2} - az_1 - \frac{1}{2} \frac{x_1^2}{D_1^2} \right) - \frac{k_2\, az_2}{D_2} \right. \right.$$

$$\left. + \frac{k_2 \left(1 - \frac{x_2^2}{D_2^2} \right)}{2D_2} \right] \left[\frac{K_1}{D_1} \left(\frac{1}{2} - bz_1 - \frac{1}{2} \frac{y_1^2}{D_1^2} \right) - \frac{k_2\, bz_2}{D_2} \right.$$

$$\left. \left. + \frac{k_2 \left(1 - \frac{y_2^2}{D_2^2} \right)}{2D_2} \right] - \frac{1}{4} \left(\frac{k_1^2 x_1^2 y_1^2}{D_1^6} \quad \frac{2k_1 k_2 x_1 y_1 x_2 y_2}{D_1^3 D_1^3} \quad \frac{k_2^2 x_2^2 y_2^2}{D_2^6} \right) \right|^{1/2}$$

We write

$$x_0' = \sqrt{\frac{2}{\pi}|A'|}\ (\chi' + H), \quad x_0'' = \sqrt{\frac{2}{\pi}|A'|}(\chi'' + H),$$

$$y_0' = \sqrt{\frac{2}{\pi}|C'|}\ (\eta' + K), \quad y_0'' = \sqrt{\frac{2}{\pi}|C'|}\ (\eta'' + K),$$

where χ', χ'' are the limits on x, and η', η'' are the limits on y in the original coordinates. Then if $J \neq 0$, eq. (12) becomes

$$I(\vec{r}_1, \vec{r}_2)\ \frac{\overline{I_0}^{\,iF'}}{D_1 D_2 J} \int_{x_0''}^{x_0'} \int_{y_0''}^{y_0'} \exp i \frac{\pi}{2} \left(\frac{A'}{|A'|} x'^2 + \frac{C'}{|C'|} y'^2 \right) dx'\, dy'$$

If x_o', x_o'', y_o', y_o'' are constants (i.e. if the projection of the surface on the xy plane is a rectangle) the integral above can be evaluated exactly, yielding

$$I(\vec{r}_1, \vec{r}_2) \approx \frac{\overline{E_0}\, e^{iF'}}{D_1 D_2 J} \left[F_{\pm}(x') - F_{\pm}(x_o'') \right] \left[F_{\pm}(y_o') - F_{\pm}(y_o'') \right] \qquad (14)$$

where $F_{\pm}(x)$ is the appropriate Fresnel integral function*. For most other boundaries, the integral can be approximated by an expression of this form since the precise shape of the boundary is unimportant unless it is circular**. When the rectangle is symmetric about the origin in the x, y variables, then $\chi' = -\chi''$, $\eta' = -\eta''$. In this case, if variations in the slowly varying quantity J are neglected, $I(\vec{r}_1, \vec{r}_2)$ has a maximum with respect to $\vec{r}_1$ (or $\vec{r}_2$) when $H = K = 0$, which occurs if $D = \frac{-k_1 x_1}{D_1} - \frac{k_2 x_2}{D_2} = 0$ and $E = \frac{-k_1 y_1}{D_1} \quad \frac{k_2 y_2}{D_2} = 0$. These equations are equivalent to Snell's law ($k_1 \neq k_2$) and the law of reflection ($k_1 = k_2$). In directions other than those given by these laws, either or both of H, K, are different from zero and become infinite as k_1 and k_2 become infinite; then the difference between at least one pair of Fresnel integrals is zero. Thus, in the limit of zero wave length the result of geometrical optics is obtained.

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* $F_{\pm}(x) = \int_0^x e^{\pm i \frac{\pi}{2} v^2}\, dv$. the signs are chosen the same as those of

$$\frac{A'}{|A'|} \quad \text{and} \quad \frac{C'}{|C'|} \quad \text{respectively.}$$

** If the boundary is a circle of radius and center at $x' = y' = 0$, the integral can be evaluated exactly and yields.

$$I(\vec{r}_1, \vec{r}_2) \approx \frac{\overline{E_0}\, e^{iF'}}{D_1 D_2 J} \; 2i \left(1 - e^{i \frac{\pi \rho^2}{2}} \right) .$$

Appendix II

Derivation of the Reflection Coefficient in Geometrical Optics

The reflection coefficient for the specular reflection of light (or very short electromagnetic waves) from a curved surface can be found exactly on the basis of geometrical optics. By exactly it is meant that no approximations, other than the assumptions of the theory, need be made. These are rectilinear propagation, the law of reflection, and conservation of energy. For definiteness the incident radiation is assumed to come from a point source although the method can be applied to any incident field.

The point source is located at $(x_1y_1z_1)$ and the point of observation at $(x_2y_2z_2)$. The coordinate system is chosen with its origin at the point on the reflecting surface from which specular reflection to $(x_2y_2z_2)$ occurs and the z axis is normal to the surface at this point. A fraction ρ^2 of the energy incident on a small area S_1 of the plane $z = 0$ around the point of reflection is reflected through a small area S_2 of the plane $z = z_2$ at the point of observation. This is a consequence of the assumptions of the theory and the assumption that the reflectivity of the surface = ρ at this point for the given direction of incidence. Therefore the ratio of the energy per unit area at the point of observation to that incident at the point of reflection is equal to $S_1\rho^2/S_2$. More precisely, the energy reflection coefficient

is the limit of this ratio as S_1 approaches zero. This limit is just ρ^2 divided by the Jacobian of the transformation which, by reflection, maps points of the plane $z = 0$ on the plane $z = z_2$. Denoting the variables on the plane $z = z_2$ by x_2', y_2', and observing that energy is proportional to amplitude squared, the amplitude reflection coefficient J is given by

$$J^2 = \rho^2 \left(\frac{\partial x_2'}{\partial x} \cdot \frac{\partial y_2'}{\partial y} - \frac{\partial x_2'}{\partial y} \cdot \frac{\partial y_2'}{\partial x} \right)^{-1}.$$

At a point on the reflecting surface $\vec{N}$, $\vec{D}_1$, and $\vec{R}$ are unit vectors, normal to the surface, along the incident ray, and along the reflected ray respectively. By the law of reflection, $\vec{R} = 2(\vec{D}_1 \cdot \vec{N})\vec{N} - \vec{D}_1$, and the equations of the ray reflected at (x,y,z) are $\frac{x' - x}{R_x} = \frac{y' - y}{R_y} = \frac{z' - z}{R_z}$.

Thus on the plane $z' = z_2$, one has $x_2' = (z_2 - z)\ \frac{R_x}{R_z} + x$;

$y_2' = (z_2 - z)\ \frac{R_y}{R_z} + y$. By the choice of coordinates the equation of the reflecting surface can be written $z = ax^2 + by^2 + \ldots$ with no xy term. The vectors N and D_1 have the components

$$N_x = \frac{-2ax}{B}, \quad N_y = \frac{-2by}{B}, \quad N_z = \frac{1}{B} \quad \text{where } B^2 = 1 + 4a^2x^2 + 4b^2y^2,$$

$$D_{1x} = \frac{x_1 - x}{D_1}, \quad D_{1y} = \frac{y_1 - y}{D_1}, \quad D_{1z} = \frac{z_1 - z}{D_1} \quad \text{where } D_1^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2.$$

Since the Jacobian is to be evaluated at the origin, the reflection coefficient becomes

$$J^{-2} = \rho^2 \left\{ 1 + z_2 \left[\frac{\partial R_x/R_z}{\partial x} + \frac{\partial R_y/R_z}{\partial y} \right] + z_2^2 \left[\frac{\partial R_x/R_z}{\partial x} \frac{\partial R_y/R_z}{\partial y} - \frac{\partial R_y/R_z}{\partial x} \frac{\partial R_x/R_z}{\partial y} \right] \right\}^{-1}.$$

The derivatives in this expression at the origin are given by

$$\frac{\partial R_x/R_z}{\partial x} = \frac{1}{z_1} - 4a \left(1 + \frac{x_1^2}{z_1^2} \right), \quad \frac{\partial R_y/R_z}{\partial x} = -4a \frac{x_1 y_1}{z_1^2},$$

$$\frac{\partial R_y/R_z}{\partial y} = \frac{1}{z_1} - 4b \left(1 + \frac{y_1^2}{z_1^2} \right), \quad \frac{\partial R_x/R_z}{\partial y} = -4b \frac{x_1 y_1}{z_1^2}.$$

Thus the reflection coefficient is given by

$$J^{-2} = \rho^2 \left\{ 1 + z_2 \left[\frac{2}{z_1} - 4a \left(1 + \frac{x_1^2}{z_1^2} \right) - 4b \left(1 + \frac{y_1^2}{z_1^2} \right) \right] + \left[z_2^2 \frac{1}{z_1^2} - \frac{4a}{z_1} \left(1 + \frac{x_1^2}{z_1^2} \right) - \frac{4b}{z_1} \left(1 + \frac{y_1^2}{z_1^2} \right) + 16ab \left(1 + \frac{x_1^2}{z_1^2} \right) \left(1 + \frac{y_1^2}{z_1^2} \right) - 16ab \frac{x_1^2 y_1^2}{D_1^4} \right] \right\}^{-1}$$

Now $(a+b) = C_m$, $4ab = G_g$, $\frac{z_1}{D_1} = \frac{z_2}{D_2} = \cos \alpha$, the mean curvature, Gaussian curvature, and cosine of the angle of incidence, all at the point of reflection on the surface, respectively. Then

$2a \frac{x_1^2}{z_1^2} + 2b \frac{y_1^2}{z_1^2} = G_{11} \tan^2 \alpha$ where G_{11} is the curvature of the surface in the plane of incidence, and the above expression for J becomes

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$$J = \rho\left\{1 + 2D_2\left[\frac{1}{D_1} - 2G_m\cos\alpha - G_{||}\sin\alpha\tan\alpha\right] + D_2^2\left[\frac{1}{D_1^2} - \frac{4G_m\cos\alpha}{D_1} + 4G_g - \frac{2G_{||}\sin\alpha\tan\alpha}{D_1}\right]\right\}^{-1/2}.$$

This expression is the reflection coefficient according to geometrical optics, and (with $\rho = 1$) is just the geometrical factor of eq. (9).

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